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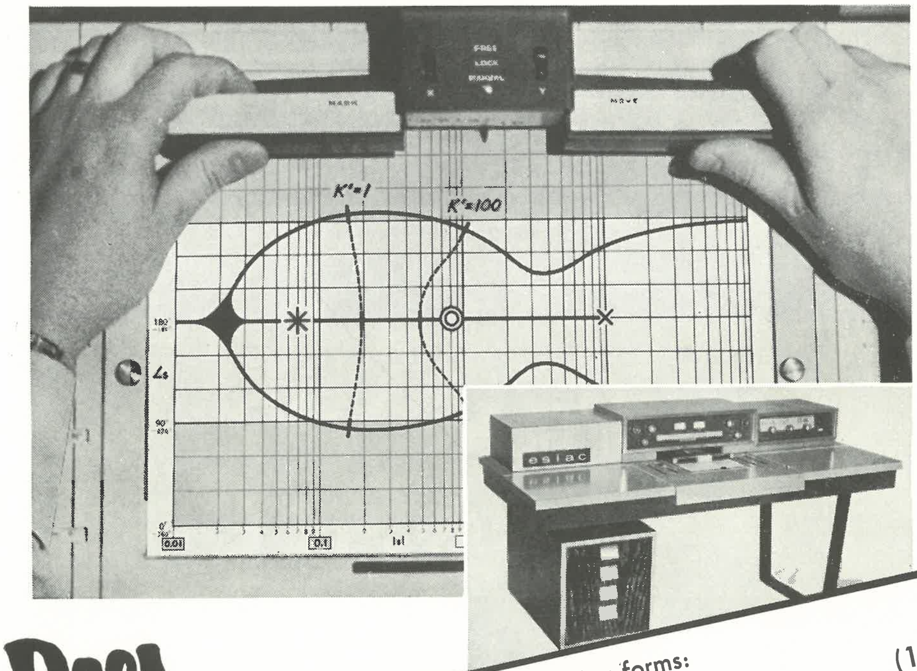
THE esiac



What it Does

The computer solves algebraic equations expressed in either of the following forms:

$$F = \pm K s^{n_0} (s - s_1)^{n_1} (s - s_2)^{n_2} (s - s_3)^{n_3} \ldots \qquad (1)$$

or

$$F = \pm K s^{n_0} \left(1 - \frac{s}{s_1}\right)^{n_1} \left(1 - \frac{s}{s_2}\right)^{n_2} \left(1 - \frac{s}{s_3}\right)^{n_3} \ldots \qquad (2)$$

where:

$F, s, s_1, s_2, s_3, \ldots$ represent vector quantities

$K, n_0, n_1, n_2, n_3, \ldots$ represent scalar quantities and can be either positive or negative

- Solves differential equations by ROOT LOCUS plots.
- Provides BODE phase and gain plots from system poles and zeros.
- Synthesizes compensation networks.
- Evaluates residues.
- Factors polynomials.
- Analyzes sampled data systems.
- Solves non-linear problems involving hysteresis or saturation by the describing function technique.
- Designs process control systems.
- Derives process transfer functions from power density spectra.
- Uses rapid, logical pole-zero programming.
- Provides permanent, plotted graphs for all stages of dynamic system design.

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For Solving
Algebraic Functions
of a
Complex Variable

by Robert L. Morgan

Lord Kelvin once said that Nature works in second order differential equations. His remark was prompted by the fact that when one sets up an equation expressing the behavior of some physical or natural phenomenon, he usually winds up with such an equation. They are usually in this form because most physical phenomena such as heat flow, the current in an electrical circuit, the flight of a missile, or the motion of vibrating members involve quantities that are changing. First derivatives are required in these equations to express rates of change, such as velocity, with second derivatives to express the rate at which velocity changes, or accelleration.

Differential equations are thus one of the most common equations the physicist or engineer is called upon to solve. Only the simplest of these have an exact solution, that is, can be solved by formula with pencil and paper. An approximate solution can usually be obtained by the method of successive approximations or other means, but such methods are so time consuming that many differential equations,such as that expressing the "three body problem" of celestial mechanics, were never solved until recently when the modern large scale digital or analog computers became available. Even with these machines, no exact solution is possible, but solutions to any desired degree of accuracy can be achieved.

There is a mathematical technique, however, that has been known for about 175 years, but has come into general use only in the last 15 years, that provides a straightforward method of solving these equations. This is the use of Laplace transforms to simplify certain mathematical operations and to convert differential, integral, and other "difficult" equations into simple algebraic equations that can be solved fairly readily.

To ease gently into the use of Laplace transforms, it would be worthwhile to draw an analogy to a transformation with which everyone is familiar, namely, the use of logarithms in arithmetic. One way of obtaining the product of two numbers is to use a table of logarithms. In its simplest form, such a table consists of two columns of numbers which are associated in pairs as in Fig. 1.

LOGARITHMIC TRANSFORM PAIRS

Original	Transform
1	0
10	1
100	2
1000	3

Figure 1

In the column labeled ORIGINAL, the original numbers appear. In the column labeled TRANSFORM are listed the numbers obtained from the original numbers by a transformation which consists of expressing the original number as a power of 10. The power, or logarithm, is the transform and the transformation is known as "obtaining the logarithm." The usual method of producing a logarithmic transformation is to enter a log table, such as Fig. 1, in the left-hand column and pass from the original number to its transform (log) in the right-hand column. This is the direct or forward transformation which is called obtaining the logarithm. If the table is used in the opposite direction, the transformation is known as the inverse log transformation, or obtaining the antilog.

To multiply two numbers, one uses a more complete table than that of Fig. 1 and proceeds from left to right to obtain the transforms (logs) of the two original numbers. The transforms are added together to obtain an intermediate result. The inverse transformation of the intermediate result is then obtained, i.e., the table is used from right to left, and the inverse transform of the intermediate result is then the product of the two numbers, or final result desired. Thus, a logarithmic transformation is used to simplify the mathematical processes of multiplication, division, evolution (raising to powers) and involution (extracting roots). A log table could also be used to add or subtract, by using the table first from right to left and then from left to right, but this substitutes the processes of multiplication or division for the processes of addition or subtraction and makes the problem harder rather than easier

The direct Laplace transformation, denoted by $\mathcal{L}$, is used in a similar way to convert differential, integral, and other equations into simpler forms. The simple equation is then solved for an intermediate function, from which the desired solution of the original equation may be obtained by the inverse Laplace transformation, denoted by $\mathcal{L}^{-1}$. The point of most of this discussion is to bring out the fact that, in practice, the direct and inverse Laplace transformations are obtained from a table of Laplace transform pairs. It is possible to calculate either transform, but one seldom does it, anymore than one calculates logarithms instead of using a log table.

After this logarithmic analogy, it will be worthwhile to point out one chief difference between logarithmic and Laplace transformations. The logarithmic transformation changes numbers into other numbers, while the Laplace transformation changes real functions into functions of a complex variable s or substitutes complex algebraic terms in s for mathematical operators, such as derivatives and integrals. Thus, an equation simplified by Laplace transforms will always be in complex form, about which more will be said later. If the original equation contains quantities in the time domain, these quantities will be transformed to the frequency domain and vice versa for the inverse transform.

An example of the use of the Laplace transformation would be the familiar equation for charging a condenser from a battery.

$$E = Ri + \frac{1}{c}\int i \, dt \qquad (1)$$

This transforms to

$$\frac{E}{s} = RI + \frac{1}{sC} \qquad (2)$$

The integral term is now gone, and we have a simple equation in terms of the complex quantity s as the unknown. Solving for the current I results in,

$$I = \frac{\frac{E}{R}}{s + \frac{1}{RC}} \qquad (3)$$

and the inverse Laplace transform of the above gives the well known solution,

$$i = \frac{E}{R} e^{-\frac{t}{RC}} \qquad (4)$$

It is not within the scope of this bulletin to go into detail on the use of Laplace transforms. Suffice it to say that they constitute a very powerful tool for breaking down certain equations into a simple form for solution, and their use is well known to those "versed in the art." The second chapter of Tom Martin's book, "Electronic Circuits," contains a very lucid discussion of these transforms and their use, and the "Table of Important Transforms" published by Hewlett-Packard Co. contains many that are particularly useful in electronic work.

Before going further in this discussion, it is advisable to define some terms which will be used from here on.

1. FACTOR - When two or more quantities multiplied together equal another quantity, then any of the first quantities are known as factors of the second quantity. Thus 2, 3, 4, and 6, are all factors of 12. Factors do not necessarily have to be whole numbers and 2.5 or 4.8 are also factors of 12.

2. ROOT - The term root has two mathematical meanings. One definition refers to one of the equal factors of a number such as the fifth root of 2 where $\sqrt[5]{2}$ multiplied by itself five times equals 2. In the other definition, roots are those values of the unknown in an equation that satisfy the equation. To use grammar school terminology, the root of an equation is the "answer" or solution one is looking for.

3. LOCUS - A locus is the path of a point that moves under certain specified conditions. Thus, a circle is the locus of a point that moves in a plane so as to maintain its distance from a fixed point, called the center, at a constant value, called the radius. Most of the curves of analytical geometry, such as elipses, parabolas, cycloids, etc. are loci of points moving under different specified conditions.

4. IMAGINARY NUMBERS - An imaginary number is one that contains an even root of a negative number. Thus $\sqrt{-4}$ is imaginary because no number exists that can be multiplied by itself to give -4. Either +2 or -2 multiplied by itself will give +4. Odd roots of negative numbers are not imaginary, but real, and $\sqrt[3]{-8}$ is an example because (-2)(-2)(-2) = -8.

5. COMPLEX NUMBERS - A complex number is one containing the sum or difference of a real and an imaginary number, such as $3 \pm \sqrt{-2}$.

6. EXPONENT - An exponent is the power to which a term is raised. Exponents may be whole numbers or fractions and either positive or negative. A fractional exponent indicates both a root and a power as in,

   $$X^{\frac{3}{5}} = \sqrt[5]{X^3}$$

   A negative exponent indicates the reciprocal of the term to the same but positive exponent as in,

   $$X^{-\frac{3}{5}} = \sqrt[5]{\frac{1}{X^3}}$$

7. ORDER - In mathematics this expression applies only to differential equations and indicates the highest order of any derivative term. Thus, the equation
$$\frac{d^2x}{dt^2} - \omega^2 x = 0$$
for simple harmonic motion, is a second order differential equation because its highest derivative term is a second derivative.

8. DEGREE - Degree applies to algebraic equations and refers to the highest exponent of the unknown in any term.

9. COEFFICIENT - A coefficient is a constant multiplier of a term, such as a and b in the equation $ax^2 + bx + c = 0$.

10. CONJUGATE - Two complex numbers are conjugate to each other if their real parts are the same and their imaginary parts are equal but of opposite sign. Thus (2+j3) and (2-j3) are complex conjugates.

Let's take a look at how simple algebraic equations are solved, starting with the quadratic, or to be more formal, the second degree algebraic equation $X^2 - 4X + 13 = 0$. Quadratics can always be solved by the quadratic formula,

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{4a}$$

where a and b are the coefficients of X^2 and X respectively and c is the constant term. Substituting these values in the formula gives,

$$X = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 2\sqrt{-9}}{2} = 2 \pm \sqrt{-9} = 2 \pm 3\sqrt{-1} \tag{5}$$

Mathematicians usually write $\sqrt{-1}$, designating an imaginary number as i and engineers write it as j. Thus, the roots of the equation are 2+j3 and 2-j3.

At first, imaginary and complex numbers, such as these, were considered as mathematical oddities, without any real physical significance. Later, however, when plotting in the complex plane became common, it was realized that multiplying by the operator j or i indicated the rotation of a vector by 90° in a counter-clockwise direction.

This is easily seen from Fig. 2 if we plot the complex number (2+j3) and successively multiply it by j, remembering that $j = \sqrt{-1}$ so that $j^2 = -1$. It is originally in quadrant I and multiplying by j the first time gives (j2-3), which is in quadrant II, exactly 90° from the original position. A second multiplication by j gives (-2-j3), in quadrant III and a third results in (-j2+3). Multiplying once more by j would put us back in quadrant I, where we started from, with (2+j3). Note that the complex plot also establishes the phase angle θ and the complex number (2+j3) could also have been expressed as a magnitude R at an angle θ, with respect to the axis of reals. The absolute value of R is, of course, determined from the right triangle relationship to be, in this case, $R = \sqrt{2^2 + 3^2} = \sqrt{13}$ and the angle θ is given by $\theta = \tan^{-1}\frac{3}{2}$.

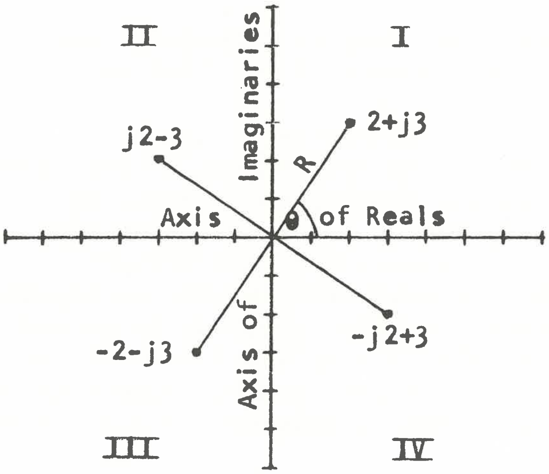


Figure 2

It will be worthwhile to bring out a third way of representing complex numbers which will be used later in this discussion. Any text on complex functions discusses Euler's equation which states that $Re^{j\theta} = R(\cos\theta + j\sin\theta)$. Fig. 3 shows that the real and imaginary components of R in the complex plane are R cos θ and R sin θ respectively. Since R sin θ is the magnitude of the imaginary component, its vector value is jR sin θ. Thus, any complex number, or point P in the complex plane, can be represented in three ways as follows:

1. $P = R\angle\theta$ ------ polar form
2. $P = Re^{j\theta}$ ------ exponential form
3. $P = a + jb$ ---- rectangular form

same complex number

In the rectangular form, a is the distance along the real axis and b is the distance along the imaginary axis, so $R = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$.

A more general method of solving the equation in the example above, or one of any degree, would be to transpose it, if necessary, to make it equal zero and then to factor it. The left-hand side of the original equation $X^2 - 4X + 13 = 0$ consists of two factors, $X-(2+j3)$ and $X-(2-j3)$ as can easily be shown by multiplying these factors together. We can, therefore, say $[X-(2+j3)]\,[X-(2-j3)] = 0$. Now the left-hand side of this equation will equal zero under two, and only two, conditions, that is, for either one of the two factors to equal zero. It is, therefore, true that if $X-(2+j3) = 0$, or if $X-(2-j3) = 0$, then, and only then, will the whole left side be zero, and the equation hold true. Therefore, each factor can be separately set equal to zero, and we find as before $X = 2+j3$ and $X = 2-j3$ are the two roots. Either value will satisfy the equation, that is, if either value is substituted for X in the equation, it will reduce to an identity, which in the case of the first root is, $4 + 4j3 - 9 - 8 - 4j3 + 13 = 0$.

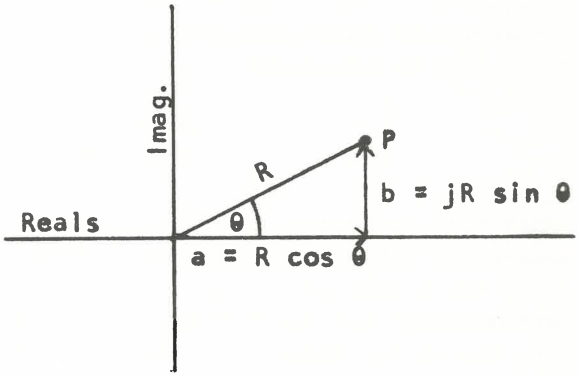


Figure 3

An equation will have as many roots as its degree, so a sixth degree equation will have six roots, for example, anyone of which is as good as the others from a mathematical standpoint because it will satisfy the equation. However, all of the roots are not necessarily of significance or interest in the problem. If the system under investigation might oscillate, then complex conjugate roots are always of interest, as such roots denote the presence of oscillation.

The discussion above shows that if a function, that equals zero, can be separated into factors, no matter how many, containing the unknown to the first power, then any factor can be set equal to zero and a solution obtained by inspection. Even if one is unable to completely factor the function so that all factors contain the unknown to only the first power, and some factors remain with higher powers, these remaining factors have been reduced in degree, possibly to the extent that if they are set equal to zero, they can be solved by some simple method. As an example, consider $X^4 + 2X^3 + 2X^2 + X = 0$ which can be factored to $X(X + 1)(X^2 + X + 1) = 0$. Thus, two roots are seen by inspection to be 0 and -1, and the other two can be simply obtained by solving $X^2 + X + 1 = 0$ by the quadratic formula.

A common method of analyzing electrical and mechanical systems is to study the response of a system to known stimuli. The stimulus is usually called the EXCITATION FUNCTION, and the response is appropriately known as the RESPONSE FUNCTION. The ratio of the response to the excitation function is known as the system TRANSFER FUNCTION. Thus,

$$\text{TRANSFER FUNCTION} = \frac{\text{RESPONSE FUNCTION}}{\text{EXCITATION FUNCTION}} \tag{6}$$

This is just another way of saying that we can get frequency response information, for example, by applying a known input and measuring the output as a function of frequency. The transfer function contains in one function all the essential information regarding the physical system. The reciprocal of the transfer function is sometimes called the CHARACTERISTIC FUNCTION from which we get,

$$\text{RESPONSE FUNCTION} = \frac{\text{EXCITATION FUNCTION}}{\text{CHARACTERISTIC FUNCTION}} \tag{7}$$

When such a system analysis is undertaken, the result as can be seen from either equation (6) or (7) is a function in fractional form. The Laplace transforms of such equations will also be in the form of a fraction whose numerator and denominator can be factored into terms containing the unknown s. Performing the factoring results in an equation in the form of,

$$F = \frac{(s-A)(s-B)(s-C)----(s-N)}{(s-a)(s-b)(s-c)----(s-n)} \tag{8}$$

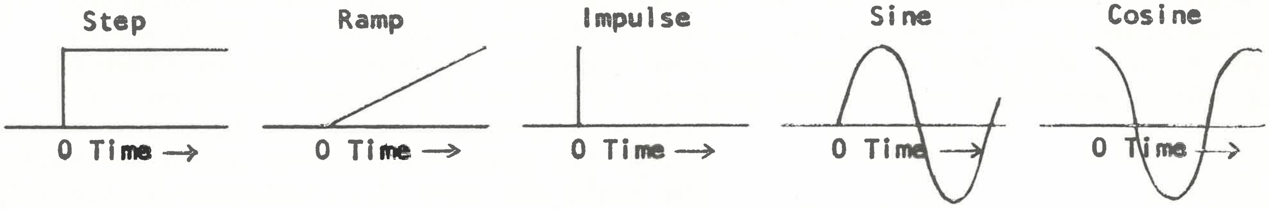
which from inspection can yield considerable information. For example, if s is equal to A, B, C, ---N, then the numerator becomes zero, and the whole function is zero. Therefore, these values of the unknown are called the zeros of the function which are usually plotted as small circles. Similarly, when s = a, b, c, ---n, then the denominator becomes zero and the whole function becomes infinity, and these values of s are, therefore, known as the poles, which are usually plotted as small X's. Thus, a value of the unknown in a function that makes it equal zero, is a zero of the function. Similarly, a value of the unknown that makes the function equal infinity, is a pole of the function. If the function is in fractional form, the zeros of the denominator are the poles of the whole function. Further, if the function equals some particular value, such as 9000, for example, then the values of the unknown that make the function equal this particular value are the roots of the equation formed by equating the function to the particular value. Finally, if the particular value mentioned above is zero, then the roots of the equation are also zeros of the function. It is a point of interest that a function will have as many poles as it has zeros, some of which may be at infinity.

When analyzing any type of system, one of the most important characteristics to be determined is the system response. As discussed on Page 6, the response is the reaction of the system to a known stimulus. The stimulus is an input action or signal which can be represented by a mathematical function, and the response is an output reaction or signal whose function is usually to be determined. Equation 6 on Page 6 can be transposed to give

$$\text{RESPONSE FUNCTION} = \text{TRANSFER FUNCTION} \times \text{EXCITATION FUNCTION} \qquad (9)$$

which states that, if the system transfer function is known, and if the excitation applied to its input is also known, then its response function can be determined merely by multiplying the transfer function by the excitation function.

A step function is frequently used as an input to determine response. The Laplace transform of a unit amplitude step occurring at time zero is $\frac{1}{s}$. Thus, if the transfer function is in Laplace form, dividing it by s will give the Laplace transform of the function representing the response to a step input. Similarly, a ramp starting at time zero has a Laplace transform of $\frac{1}{s^2}$. The transfer function itself is also a response function, as it can be considered as the transfer function multiplied by unity and unity is the Laplace transform of a unit impulse (spike of infinite height and zero width). Some of these common excitation functions are shown in Fig. 4.



Common Excitation Functions

Figure 4

The response of a system (or portion of a system) is usually considered to be in two parts. The first in order of time, is the transient response to a given input, and it is concerned with nonrepetitive starting transients, occurring immediately after the application of the excitation. These die out eventually in a stable system. The second is the steady state response that exists after the starting transients have died out. Frequency response, which is the steady state response to sinusoidal inputs, can be obtained from a transient analysis by multiplying the transfer function by

$$\frac{\beta}{s^2+\beta^2} \tag{10}$$

for sine wave, or

$$\frac{s}{s^2+\beta^2} \tag{11}$$

for cosine excitation, to obtain the response function and then, by standard methods, evaluating the resulting function at time infinity, when transients have died out and only steady state response is left.

In many cases, it is only necessary to know the nature of the transient response of a system without regard to the relative magnitudes of its various components. For such information it is only necessary to examine the poles of the transform of the response function and note their location in the complex s plane. For example, if the poles are real, that is, lie only on the real axis, then the response is non-oscillatory. Oscillation is present only when complex conjugate poles are present. If the pairs of complex conjugate poles lie on the imaginary axis, the oscillation is undamped. If they lie off of this axis in the left half of the s plane, then the response is an exponentially damped oscillation. If they lie off of the imaginary axis in the right half of the plane, then the response is an exponentially growing oscillation or runaway condition. Thus, a "physically realizable complex function" can have poles only in the left half of the complex plane. This is illustrated in Fig. 5, where the pairs of poles plotted, represent conditions as follows:

X_1 and X_2-------------------------runaway
X_3 and X_4-------------------------undamped
X_5 and X_6-------------------------exponentially damped
X_7 and X_8-------------------------critically damped
X_9 and X_{10}-------------------------over critically damped

The movement of the pairs of poles in the complex plane, as the damping is varied, is also illustrated on Fig. 5. At minimum damping the poles are at positions such as X_1 and X_2. As damping is increased, they move to the left to X_3 and X_4, X_5 and X_6 etc. With more damping they move together and superimpose as shown by $X_{7\text{-}8}$, which represents a 2nd order pole and a critically damped condition. A further increase in damping, beyond critical, causes the poles to separate again and move to the right and left as illustrated by X_9 and X_{10}. Thus, the nature and positions of the poles present a wealth of information and provide a quick method of qualitative but not quantitative analysis of the transient response of a system.

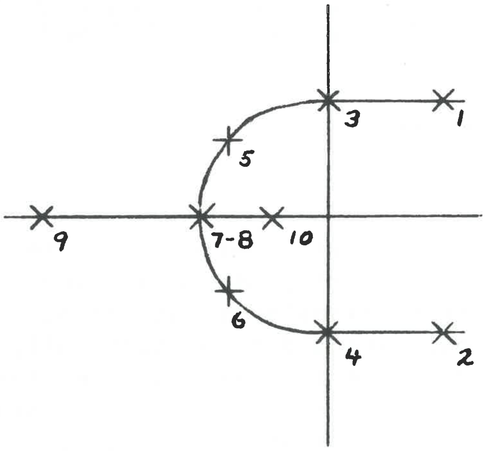


Figure 5

If quantitative information regarding the transient response is needed, then equation (8) can be broken up into partial fractions whose coefficients, or numerators, represent the magnitudes of the response terms. Equation (8) can thus be rewritten to the form

$$F = \frac{D}{(s-a)} + \frac{E}{(s-b)} + \frac{F}{(s-c)} + \text{---} \frac{K}{(s-n)} \tag{12}$$

where the values of the terms D, E, F, ---K in the numerators are, as yet, undetermined but which will be given values which will make equation (12) equal to equation (8). As a numerical example consider

$$F = 2 = \frac{80}{(2)(4)(5)} = \frac{D}{2} + \frac{E}{4} + \frac{F}{5} \tag{13}$$

If in equation (13) we give D, E, and F the values 1, 2, and 5 respectively, then the right half will equal the left half, and we will have converted the product of three terms into the sum of three different terms. Using well-known standard procedures, the partial fraction coefficients, D, E, F, ---K, in the numerators of (12) can be evaluated to convert any equation in the form of (8) into a partial fraction expansion such as (12).

Assuming that the function is in the Laplace form, then each partial fraction in the function can be converted by the inverse Laplace transformation to a function which expresses the contribution of that term to the over-all transient response. The over-all response is then the sum of the individual responses contributed by each term of the partial fraction expansion. As an example, if the partial fractions were

$$\frac{1}{s+\alpha} + \frac{\beta}{s^2+\beta^2} \tag{14}$$

the inverse transform would give

$$e^{-\alpha t} + \sin \beta t \tag{15}$$

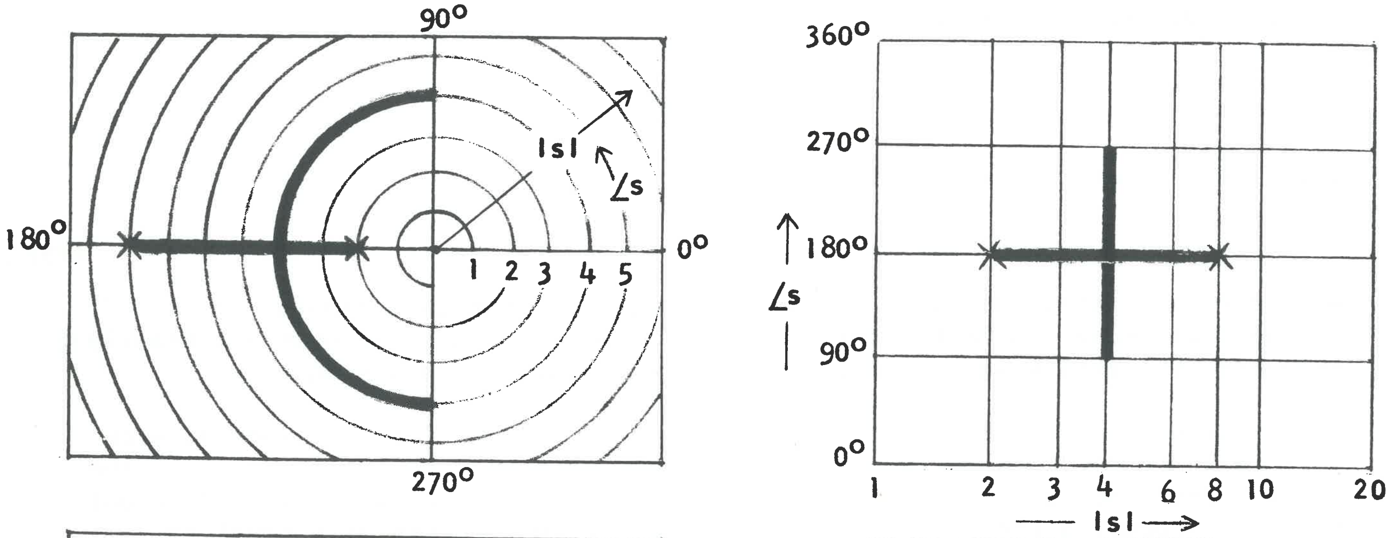
An experienced servo designer would know that the first term denotes an exponential decay, while the second is a sine wave so that the over-all transient response is a sine wave superimposed on an exponentially decaying axis.

The procedure for evaluating the coefficients of the partial fractions at each pole is known mathematically as "obtaining the residues" of the function at the poles. The name is an apt one, as it is done in simple cases by cancelling out the term producing the pole and then setting the coordinates of s equal to those of the pole. The value of the remaining portion of the function is then the residue,or the coefficient of the partial fraction at that pole.

Before going further it would be worthwhile to discuss the Root Locus Method of evaluating a function. This method is a graphical means of obtaining the roots of the characteristic equation or transfer function of a system. The method was fathered by Walter R. Evans of North American Aviation some years ago, but it is only in the last two or three years that it has found wide acceptance. The Root Locus Method treats each term of an equation as a vector, which, of course, has a magnitude and an angle. The equation describing the system characteristic is factored into a series of vector products, which means the product of their magnitudes and the sum of their angles. This equation is then separated into two equations, the first consisting of the magnitude products, and the second the sum of the angles. The poles and zeros of the equation are then plotted in the complex s plane and used as a basis for sketching the locus of the roots as loop gain (K) is varied. The knowledge that the loci of constant phase always run from poles to zeros or vice versa is of considerable help in sketching these in. Any root, then, lies on the root locus at a position determined by the loop gain. Because there is both a magnitude and a phase equation, there will be two root loci, one the locus of roots of the phase equation as phase is held constant (usually at 180°), and the other the locus of roots of the magnitude equation as the magnitude is held constant. Roots, satisfying both equations, lie at the intersection of the two loci.

The Spirule was developed as an aid in plotting root loci. It is essentially a pivoted disk which can be rotated to add angles in succession. It is also equipped with a log scale that permits multiplying magnitudes by adding their logs somewhat as is done with a slide rule.

People who use the Root Locus Method are used to making their root locus plots in polar form, while the ESIAC plots them in rectangular form, using a horizontal scale of log magnitude and a vertical scale of linear angle. The two types of plot are shown in Fig. 6, where the same root loci are plotted in both cases.



s Plane

Log s Plane

Figure 6

In the polar form it is easily seen that one locus is of varying magnitude at a constant phase angle of 180°, and the other is of varying phase at a constant magnitude of 4, with poles plotted at $8\angle 180^\circ$ and $2\angle 180^\circ$. The ESIAC form of the plot at the right presents exactly the same information in rectangular form, but it may take people who have been working with the polar plot a little time to become used to the rectangular form. There is an angle scale shift switch on the ESIAC which changes the phase angle scale from that shown in the right half of Fig. 6 to one with zero at the horizontal center line with -90° and -180° major lines below it and +90° and +180° major lines above it. Note that the top and bottom lines of either of these angle scales are identical, so that the plot in both cases is equivalent to having the paper wrapped around a drum having a horizontal axis with the 0 and 360° lines (or the -180° and +180° lines on the other scale) superimposed. Note also that physically realizable functions, that must lie in the left half of the polar plot, must lie in a band between +90° and +270° (or, with the other scale, outside of the band between +90° and -90°) on the ESIAC log s plot.

As described in the ESI literature, the ESIAC will accept linear equations in factored form. Two general types of equation are acceptable, one

$$F = K\, s^{n_0}(s-s_1)^{n_1}(s-s_2)^{n_2}----(s-s_i)^{n_i} \tag{16}$$

and the other

$$F = K\, s^{n_0}\left(1-\frac{s}{s_1}\right)^{n_1}\left(1-\frac{s}{s_2}\right)^{n_2}----\left(1-\frac{s}{s_i}\right)^{n_i} \tag{17}$$

K can be either plus or minus, as can any exponent and all terms are complex. Design problems can readily be set up in either form, although one form may require less preparation than the other, depending on the problem. For the purposes of the following discussion, only the $(s-\bar{s}_i)$ form will be considered. It should be apparent that either form may be fractional. As an example consider the $(s-s_i)$ form with every other exponent negative as follows:

$$F = K\, s^{n_0}(s-s_1)^{-n_1}(s-s_2)^{n_2}(s-s_3)^{-n_3} \tag{18}$$

Inasmuch as negative exponents designate reciprocals, we then have

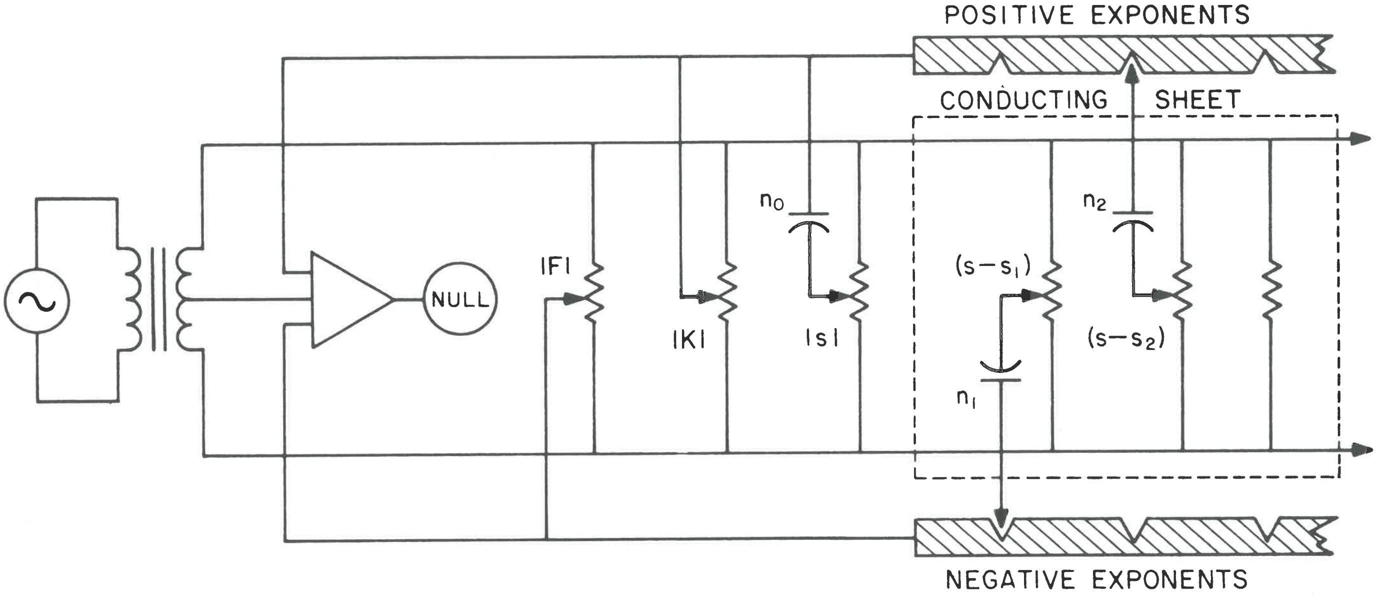
$$F = \frac{K\, s^{n_0}(s-s_2)^{n_2}}{(s-s_1)^{n_1}(s-s_3)^{n_3}} \tag{19}$$

which is obviously fractional.

The basic principle of operation of the ESIAC is that each term in the equation is represented by a current. When the computer is manipulated so that an equality between currents is indicated by a null meter, then there is a corresponding equality between mathematical terms and a solution is achieved. However, note one very important point. ESIAC equations (16) or (17) are in the form of products of terms, and it is not easy to multiply currents. To get around this and simplify the ESIAC, Dr. Merle Morgan has resorted to an ingenious dodge. This is to operate with the log of the equation instead of its normal form. The log of a complex number is the log of its magnitude, plus j times its angle. If in addition, the complex number has an exponent, then this exponent multiplies both the log magnitude and the angle. Thus, if the log of equation (16) is taken, each complex term splits into two terms, one involving log magnitude and the other involving angle. The magnitude and angle portions of the log of equation (16) can then be separated into two equations as follows:

$$\left.\begin{aligned} \log|F| &= \log|K| + n_0\log|s| + n_1\log|s-s_1| + n_2\log|s-s_2| + --n_i\log|s-s_i| \\ j\angle F &= j\left[\angle K + n_0\angle s + n_1\angle(s-s_1) + n_2\angle(s-s_2) + --n_i\angle(s-s_i)\right] \end{aligned}\right\} \tag{20}$$

The original equation is now replaced by a pair of equations, one expressing an equality of magnitudes, and the other an equality of angles, with all terms in either equation added together instead of multiplied together. Note that, if in the ESIAC, we use currents that are proportional to the log of the currents originally discussed we can now add them instead of multiplying them to equate one side of the equation to the other in a null indicator. Because there are two separate equations to solve, the ESIAC really consists of two separate units, one to obtain a solution of the magnitude equation, and the other to solve the phase equation. The plotting mechanism and scales for s are common to both.



$$\boxed{1 \log |F| = 1 \log |K| + n_0 \log |s| - n_1 \log (s-s_1) + n_2 \log (s-s_2)}$$

Figure 7

Fig. 7 is a simplified schematic of only the magnitude balance section, where the conductive sheet is represented by a group of potentiometers whose sliders symbolize the probes. The currents for log |F|, log |K|, and log |s| are obtained from a power supply through calibrated potentiometers adjusted by sliders. The currents for terms of the form $(s-s_i)$ are obtained, via the probes, from potentials set up on a conductive sheet, which at present is Teledeltos paper but which later may be replaced by glass or plastic with a uniform resistive coating. The analogs of the exponents of each term are capacitors in series with each corresponding current source. For simplicity of construction, the ESIAC potential sources are obtained from a 120 volt 60 cycle AC source, but DC sources could have been used just as well.

It may help in understanding the ESIAC if an example is given to show how the currents, and hence the mathematical terms are equated. Assume that the exponents of the original equation are such that the signs of the magnitude equation of (20) are as shown below:

$$\log |F| = \log |K| + n_0 \log |s| - n_1 \log |s-s_1| + n_2 \log |s-s_2| \qquad (21)$$

If the negative term on the right is transposed to the left, we then have

$$\log |F| + n_1 \log |s-s_1| = \log |K| + n_0 \log |s| + n_2 \log |s-s_2| \qquad (22)$$

where all terms on each side are of the same sign, namely positive. Then currents representing the left-hand terms can be run to one side of a null meter and currents representing the right side can be run to the other side of the same meter. If the currents representing the variables are adjusted for a null balance by moving the probes, it then follows that since these currents balanced the meter, the values of the variables corresponding to these currents will balance the equation. Thus, these values are roots of the magnitude equation. Similar nulls on the phase angle meter establish angles that balance the angle equation and simultaneous nulls on both meters permit reading from the computer scales (or plotting on graph paper) values of both magnitude and angle, constituting complex numbers that are roots or solutions to the original equation.

One might well ask why a negative current could not have been used for the negative term of equation (21) and thus avoid introducing this current into the opposite side of the null meter, along with log $|F|$. The reason is, that to simplify construction and programming, only one conductive sheet is used for magnitude and one for phase. If the sheet were connected to the power supply with a polarity to provide negative currents, then necessary positive currents could not be obtained from the same sheet and vice versa. Thus, all $(s-s_i)$ currents come from one sheet, regardless of the sign of the mathematical term, and this means that currents representing terms of opposite sign must go to opposite sides of the null indicator.

Fig. 8 shows how the power source is connected to the magnitude sheet and the resulting lines of current flow for solving equations in the $(1-\frac{s}{s_i})$ form. Fig. 9 shows the same thing for the phase sheet. Note that the phase sheet is split in half horizontally to form two sections, insulated from each other, and supplied from a center tapped power source. In these figures the current lines on the magnitude sheet are the equipotential lines on the phase sheet and vice versa. For the $(s-s_i)$ form of the equation, the connections to the conductive sheets are made in a slightly different manner which will not be described here.

As will be seen in the picture of the ESIAC on Fig. 10, its controls consist of a vertically mounted group in the center, back of the plotting board, two potentiometers on each side of the plotting board, and a vertically mounted group of controls to the right and rear of the plotting board.

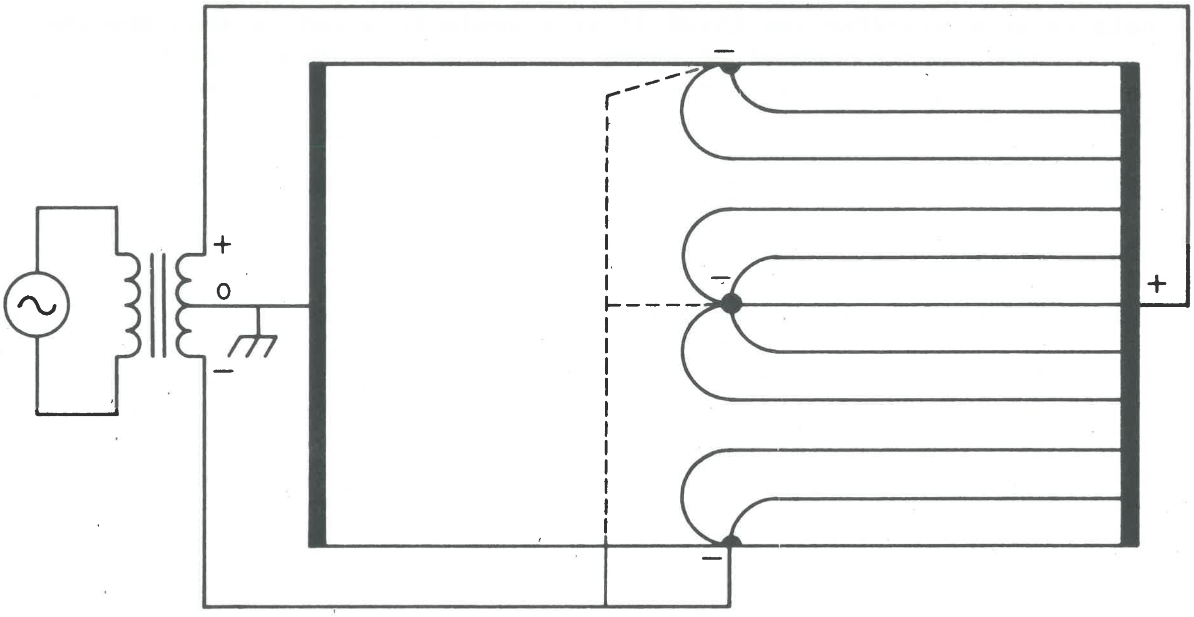


Figure 8

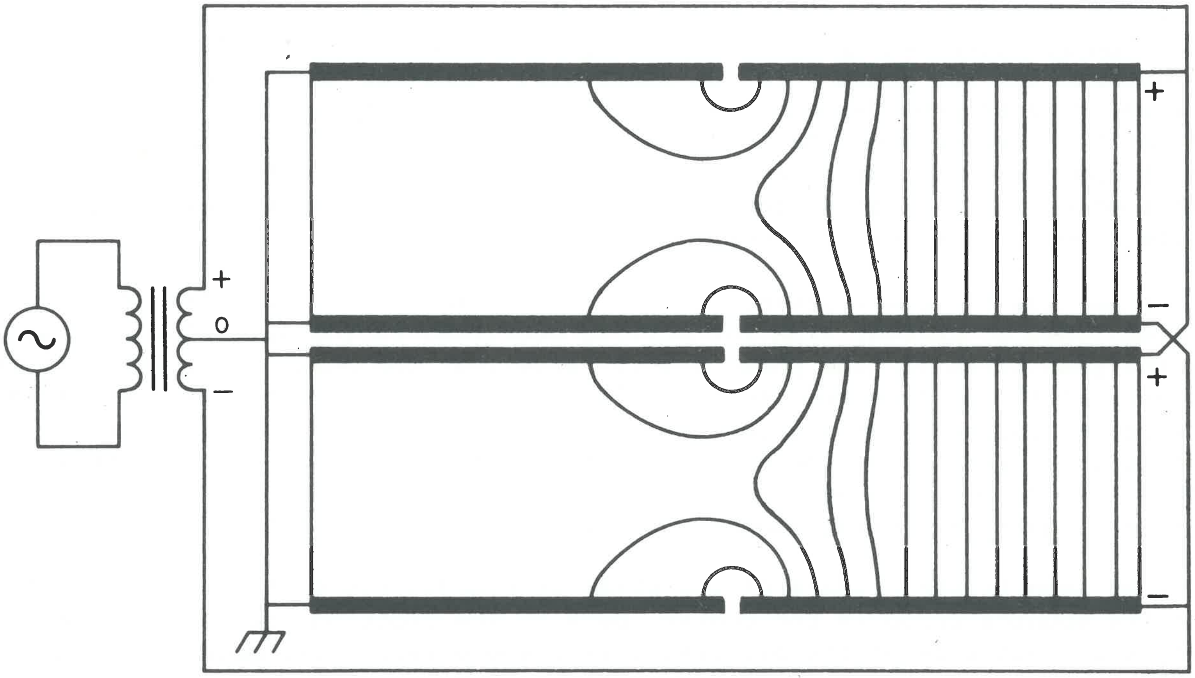
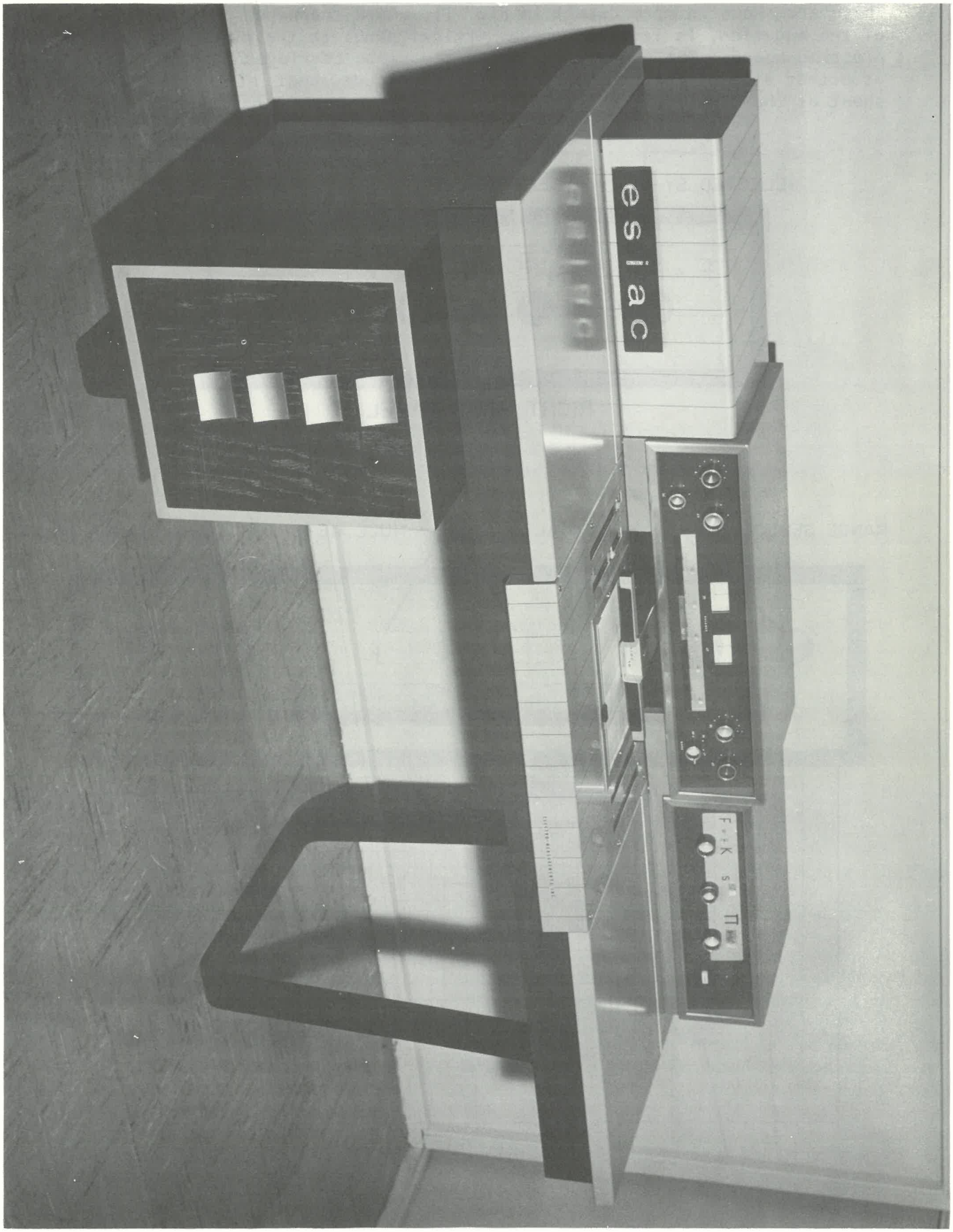
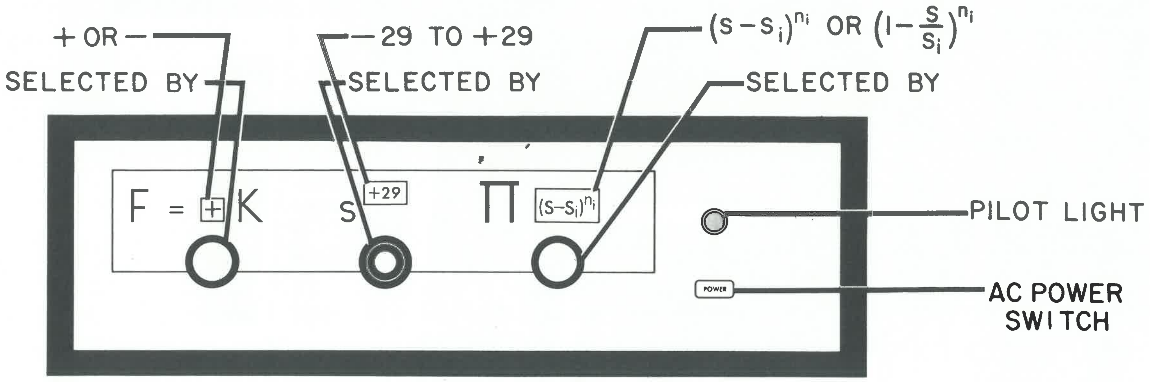


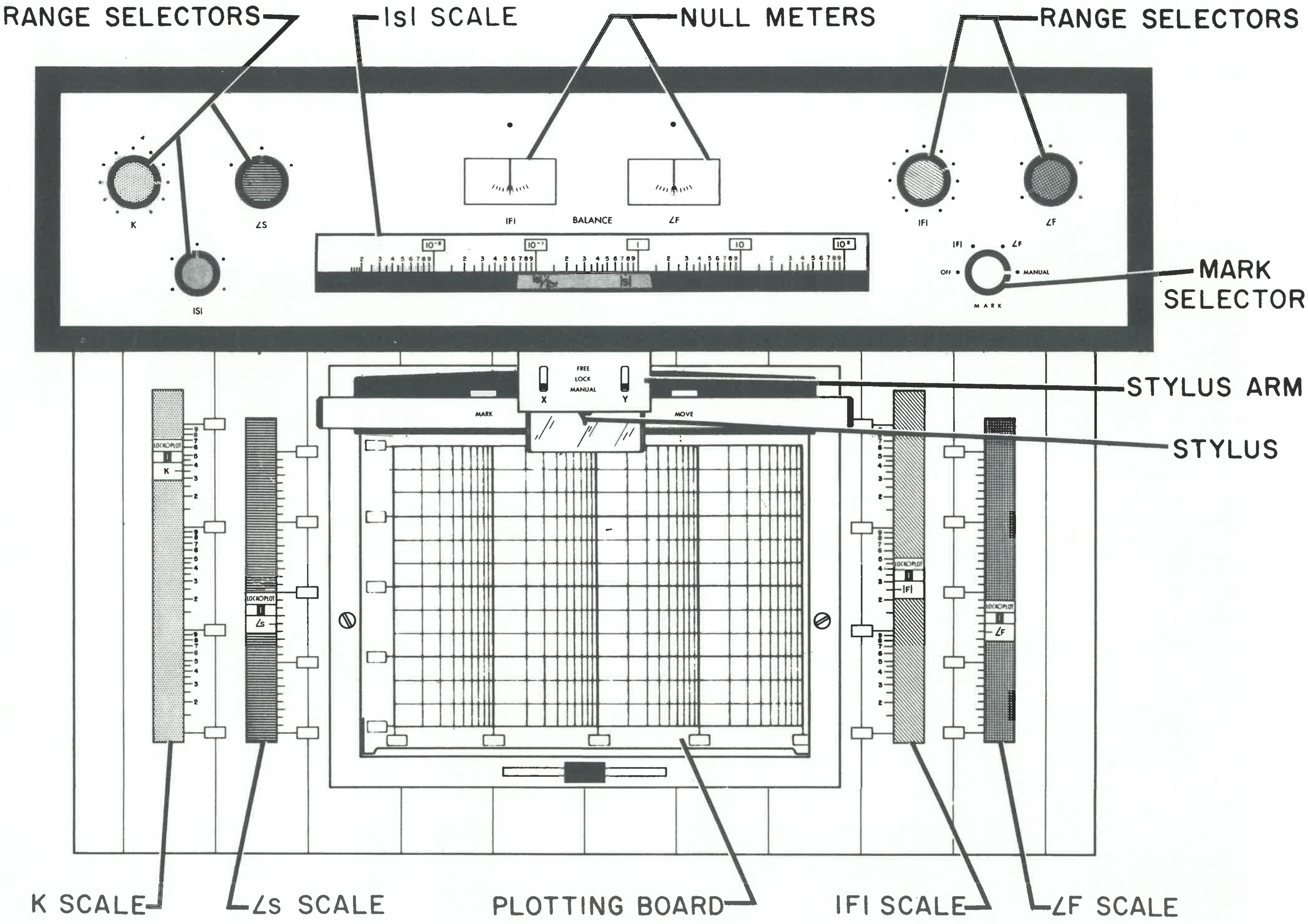
Figure 9


esiac

These are shown in more detail in Fig. 11, where the upper portion, giving the form of the equation, is located on the vertical panel to the right and rear of the plotting board. The conductive sheets and probe carriages are under hinged covers at either side of the plotter, with the magnitude sheet at the left and the phase sheet at the right.



RIGHT HAND PANEL



CENTER PANELS

Figure 11